

SL(2,Z) duality of Born-Infeld theory from non-linear self-dual electrodynamics in 6 dimensions

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Abstract

We reformulate the Born-Infeld action, coupled to an axion and a dilaton in a duality manifest way. This action is the generalization of the Schwarz-Sen action for non-linear electrodynamics. We show that this action may be obtained by dimensional reduction on a torus of a self-dual theory in 6 dimensions. The dilaton-axion being identified with the complex structure of the torus. Applications to M-theory and the self-dual IIB three brane are investigated.

Introduction

Recently, there has been much interest in the role of dualities in string and field theories. Usually, these duality symmetries are not manifest in the action but are seen as symmetries of the equations of motion together with the Bianchi Identities. In the most elementary example, of 4-dimensional electromagnetism in the absence of sources, the equations of motion and the Bianchi Identities swap roles, hence electric and magnetic fields, are exchanged. Specifically, $E \rightarrow B$ and $B \rightarrow -E$. Yet, the action is not invariant under such a transformation. With the contemporary understanding of the importance of duality, it is desirable to incorporate these hidden duality symmetries directly into the action. In [1], Schwarz and Sen describe how to form duality invariant actions for, amongst others, the Maxwell action and the low energy effective action of heterotic string theory compactified on a six torus. An essential ingredient in their formulation was to give up manifest Lorentz invariance. This was inspired by the work of various people [2] in writing down the action for self-dual field theories. Later, it was shown that manifest Lorentz invariance may be recovered by including in the action an auxiliary field that allows some non trivial gauge symmetries. This is the so called PST formalism [3].

One of the most major developments in recent years is the introduction of D-branes in string theory [4]. It has been shown that the world volume action of a D-brane is of Born-Infeld type [4,5]. In particular, the bosonic D-3 brane will have a world volume action that is the Born-Infeld action. It has been shown, [5,6] that this action has a duality symmetry of a similar type to the Maxwell theory. Under world volume duality transformation of the vector fields in the brane, the form of the action is left invariant but the background axion-dilaton fields are inverted. This may be extended to give a full $SL(2,R)$ duality once shifts to the axion field are taken into account. Of course we only expect the duality group to be $SL(2,Z)$ for the quantum theory.

We shall describe how one may produce the 4-dimensional Born-Infeld action associated with the 3-brane from self-dual non-linear 2-form electrodynamics in 6-dimensions, via dimensional reduction on a torus. This is the theory associated with the 5-brane in M-theory [7]. In fact it is this theory, which has only been recently described, that will prove vital to our construction. Moreover, we will in the process reformulate the Born-Infeld theory in such away that the duality symmetry is manifest- in a generalization of the Schwarz-Sen construction discussed above. Importantly, the duality in the four dimensional theory will be shown to be a consequence of the geometry of the torus. That is we shall be able to identify the coupling in the theory with the complex structure of the torus. Duality then results from transformations of the complex structure that leave the torus invariant ie $SL(2,Z)$. This is in

the spirit of [8] where duality in lower dimensional theories is seen as a geometrical property of the compact space used in dimensional reduction. Of course, the context for this work is in M-theory. The Supersymmetric version of these actions should be associated with the M-theory 5-brane and the IIB self-dual 3-brane. In that context, the $SL(2, \mathbb{Z})$ symmetry is an S-duality in IIB string theory. The 3-brane is left invariant but the fundamental string and D-string form a doublet along with the dilaton and axion. Future work will concentrate on these M-theoretic applications which require the full supersymmetric actions and also the possibility of including other M-theory modes such as those from the membrane.

The structure of the paper is as follows. We will begin by reviewing the action for Born-Infeld coupled to an axion and dilaton and the resulting duality symmetry. We then describe how this action may be obtained from non linear self dual 2-form electrodynamics in 6 dimensions. We then demonstrate how by dropping Lorentz invariance we may also construct an action for the Born-Infeld theory that duality symmetry manifest, providing a Schwarz Sen type action for Born-Infeld. Finally, we discuss how this fits into the brane picture.

Born-Infeld action and duality

We begin with the Born-Infeld action with the coupling to a background dilaton, ϕ and axion, C .

$$S = \int_{M^4} d^4x \sqrt{-\det(\eta_{\mu\nu} + e^{-\phi/2} F_{\mu\nu})} + \frac{1}{8} i \epsilon^{\mu\nu\rho\sigma} C F_{\mu\nu} F_{\rho\sigma} \quad (1)$$

We will introduce a complex field $\lambda = C + i e^{-\phi}$ that will prove useful later. η is the metric and F the field strength of an abelian vector potential A , defined as usual by $F = dA$. We will work in flat space time with $\eta = \text{diag}(1, -1, -1, -1)$. We performing the duality transformation on the above action by the usual process [5]. That is treat F as a generic two form and then add a term to the lagrangian that imposes F be closed via a lagrange multiplier. We can then integrate out F in favor of the lagrange multiplier in the path integral, or classically, solve the equation of motion for F in terms of the lagrange multiplier and then substituting into the lagrangian. The duality transformed action is, in terms of dual gauge field strength denoted with a tilde:

$$S_D = \int_{M^4} d^4x \sqrt{-\det(\eta_{\mu\nu} + \frac{1}{\sqrt{e^{-\phi} + e^{\phi} C^2}} \tilde{F}_{\mu\nu})} + \frac{1}{8} i \epsilon^{\mu\nu\rho\sigma} \frac{-C e^{\phi}}{e^{-\phi} + e^{\phi} C^2} \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma} \quad (2)$$

If we define the dual dilaton axion as follows:

$$e^{-\tilde{\phi}} = \frac{1}{e^{-\phi} + e^{\phi} C^2}$$

and

$$\tilde{C} = \frac{-Ce^\phi}{e^{-\phi} + e^\phi C^2}$$

Hence, the dual of λ becomes:

$$\tilde{\lambda} = \frac{-1}{\lambda} \quad (3)$$

The action (2) is then identical to (1) for the dual fields. If we think of the background dilaton axion as playing the role of an effective gauge coupling constant we have the following picture. The form of the action is left invariant but the coupling is inverted. This is like the usual S-duality. There is also the usual allowed integer shifts in C that leave the path integral invariant (strictly speaking we should normalize our action by factor of 4π). Together the two transformations generate the full $SL(2, \mathbb{Z})$.

As has been shown in a variety of contexts [8], duality related theories can be obtained by compactification of a single parent theory in higher dimensions. In the case of Maxwell theory in 4-dimensions, S-duality was derived by compactifying a self dual 2-form theory Maxwell theory in six dimensions. (By the Maxwell theory, we mean a theory with an action that has the usual field strength squared as opposed to the the Born-Infeld type theories which have a more complicated non-linear action.) To generalize this idea to the case of Born-Infeld however, we require that the six dimensional, parent theory also be non-linear and self dual. The Lorentz invariant form for such a theory was not known until very recently. Schwarz and others [7] in the context of searching for the M-theory five brane action produced the action from which we will take our starting point.

$$S = - \int_{M^6} d^6x \sqrt{-\det(G_{\mu\nu} + i \frac{\tilde{H}_{\mu\nu}}{\sqrt{(\partial a)^2}})} - \frac{\tilde{H}^{\mu\nu} H_{\mu\nu\rho} \partial^\rho a}{4(\partial a)^2} \quad (4)$$

$G_{\mu\nu}$ is the metric in six dimensions. We will introduce some form notation that will be useful later. The space of p-forms that have values on a d dimensional manifold, M^d is called $\Lambda^p(M^d)$. So, for the fields in the above action: $\tilde{H} \in \Lambda^2(M^6)$, $a \in \Lambda^0(M^6)$. We define \tilde{H} by the following:

$$\tilde{H} = *(H \wedge da) \quad (5)$$

Where $*$ is the Hodge dual acting in 6 dimensions. $H \in \Lambda^3(M^6)$ is the field strength of the abelian potential $B \in \Lambda^2(M^6)$ defined by the usual relation $H = dB$.

The field a is completely auxiliary. However, it is required to preserve Lorentz invariance in the action. We will not discuss all the properties of this action here but refer to the literature [7]. However, there are two symmetries that will prove relevant. One is the usual gauge symmetry for an abelian potential, $\delta B = d\chi$. The other is the non trivial gauge symmetry introduced by the new auxiliary field :

$$\delta B = \psi \wedge da \quad (6)$$

where $\psi \in \Lambda^1(M^6)$ is the gauge parameter.

Dimensional Reduction

Now, we will double dimensionally reduce this action on a torus, keeping only the zero modes. So we carry out the following: $M^6 \rightarrow M^4 \times T^2$. We make the following ansätze,

$$G = \eta \oplus \pi \quad (7)$$

where π is the metric on the torus and η the metric in four dimensions. This is in fact a truncation (consistent) where we do not consider the possible Kaluza-Klein fields corresponding to the compact dimensions, of which there should be two. Our ansatz for the gauge field B is again truncated. We have only included a part that couples to the conformal part of the torus. Hence,

$$B = A^I \gamma_I \quad \Rightarrow \quad H = F^I \gamma_I \quad (8)$$

where $A^I \in \Lambda^1(M^4)$, $F^I = dA^I$ and γ_I are the canonical one forms associated with the non trivial homology one cycles on the torus. Hence, they form a basis for $H^1(T^2, \mathbb{Z})$. There are two such one cycles, hence $I = 1, 2$. Note, $d\gamma_I = 0$ and $d^* \gamma_I = 0$.

Now we have two natural possibilities for the auxiliary field a . It can be chosen such that $da \in \Lambda^1(T^2)$ or $da \in \Lambda^1(M^4)$. We will look at the consequences of both choices. Though of course, both possibilities must be physically equivalent. In the first instance, we find the following for \tilde{H} :

$$\tilde{H} = *F^{I*}(\gamma_I \wedge da) \quad (9)$$

where the Hodge star in front of F acts in M^4 and the Hodge star in front of the parentheses acts in T^2 . This gives $\tilde{H} \in \Lambda^2(M^4)$. We can now factorize the determinant, using $\det(A \oplus B) = \det(A)\det(B)$. So that

$$S = \int_{M^4} \int_{T^2} -\sqrt{\pi} \sqrt{-\det(\eta_{\alpha\beta} + i \frac{*F^I_{\alpha\beta} *(\gamma_I \wedge da)}{\sqrt{(\partial a)^2}})} - \frac{*F^{I\mu\nu} F^J_{\mu\nu} *(\gamma_I \wedge da)^*(\gamma_J \wedge *da)}{4(\partial a)^2} \quad (10)$$

To investigate this action we will now make a gauge choice for da . A natural choice is to take $da \in H^1(T^2, \mathbb{Z})$. So suppose we choose da to be γ_L . The local symmetry (4) then allows us to gauge away one of the fields, A^L . It only remains to evaluate the terms in the action such as $\gamma_L \wedge \gamma_I$ and $\gamma_L \wedge * \gamma_I$. We can evaluate these using an explicit basis for $H^1(T^2, \mathbb{Z})$. These terms are proportional to the volume form Ω as follows:

$$\gamma_I \wedge^* \gamma_J = \frac{M_{IJ}\Omega}{\mathcal{V}}, \quad \gamma_I \wedge \gamma_J = \frac{L_{IJ}\Omega}{\mathcal{V}} \quad (11a)$$

where $\mathcal{V} = \int_{T^2} \Omega$ and M_{IJ} and L_{IJ} are the period and intersection matrices defined as follows:

$$M = \int_{T^2} \begin{pmatrix} \gamma_1 \wedge^* \gamma_1 & \gamma_1 \wedge^* \gamma_2 \\ \gamma_2 \wedge^* \gamma_1 & \gamma_2 \wedge^* \gamma_2 \end{pmatrix} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad L_{IJ} = \int_{T^2} \gamma_I \wedge \gamma_J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (11b)$$

Hence, substituting in (11) into (10) and integrating over the torus we find the action:

$$S = \int_{M^4} -\sqrt{-\det(\sqrt{\mathcal{V}}\eta_{\alpha\beta} + i^*F_{\alpha\beta}\omega)} - {}^*F^{\mu\nu}F_{\mu\nu}\rho \quad (12)$$

Where ω and ρ depend on the specific choice of da . The two independent choices for da give the following:

$$\begin{aligned} da = \gamma_1 &\Rightarrow \omega = \sqrt{\tau_2}, \quad \rho = \tau_1 \\ da = \gamma_2 &\Rightarrow \omega = \sqrt{\frac{\tau_2}{|\tau|^2}}, \quad \rho = \frac{-\tau_1}{|\tau|^2} \end{aligned} \quad (13)$$

Redefining, $F' = i^*F$ and rescaling the metric as follows $\eta'_{\alpha\beta} = \sqrt{\mathcal{V}}\eta_{\alpha\beta}$ allows us to identify the action (12) with the Born-Infeld action given in (1). With this identification we then compare the action (12) for different choices of da with the actions (1) and (2). For choice $da = \gamma_1$ we identify (12) with (1) and for $da = \gamma_2$ we identify (12) with the dual theory (2). These identifications imply simply that we must identify the dilaton-axion with the complex structure. That is,

$$\lambda = \tau \quad (14)$$

The duality transformation that inverts λ is then given by making a different choice for da . Hence, we see how duality becomes a gauge symmetry of this theory.

The other possibility mentioned above is that $da \in \Lambda^1(M^4)$. Note, that once such a choice is made, manifest Lorentz invariance is broken as da picks out a direction in space time. We will go immediately to the obvious choice $da = dt$. Other choices for da , will not be related by duality as in the previous case but by Lorentz transformations. We use the same ansatz as before for the metric and the two form gauge field B , however now we find that the matrix, $G + i\tilde{H}$ does not decompose into block diagonal form and so the determinant will not immediately factorise. Hence, we explicitly expand out the determinant using the following identity (where $H_{\mu\nu}$ is an antisymmetric tensor in 6 dimensions of rank 4):

$$\det(G_{\mu\nu} + iH_{\mu\nu}) = \det G \left(1 + \frac{1}{2}tr H^2 + \frac{1}{8}(tr H^2)^2 - \frac{1}{4}tr H^4\right) \quad (15)$$

We now define the magnetic and electric field strengths in the usual way: $B_i = \frac{1}{2}\epsilon_i^{jk}F_{jk}$ where $i, j = 1, 2, 3$ and $E_i = F_{i0}$. We now substitute in the E and B fields into the action expanded

out using the above identity. We also have used the period and intersection matrices of the torus as before and integrated over the torus. We also make the same scaling of the metric so to absorb the area dependence into the metric.

$$S = \int_{M^4} d^4x \sqrt{-\eta'} \sqrt{-[1 + B_i^I B^{Ji} M_{IJ} + \frac{1}{2} B_i^I B^{Ji} B_j^L B^{Kj} M_{IJ} M_{LK} - \frac{1}{2} B_i^I B^{Li} B_j^J B^{Kj} M_{IJ} M_{LK}] + E_i^1 B^{2i} - E_i^2 B^{1i}} \quad (16)$$

So note, this action with two magnetic fields is symmetric in B^1 and B^2 . (Obviously, a choice of space-like da would give a pair of electric fields.) These fields are related to each other by duality, as we will show when we demonstrate the equivalence of the above action to Born-Infeld. It is this action that we claim is the Born-Infeld equivalent of the Schwarz-Sen action for Maxwell theory. (Recently, several people, using very different approaches to those described here, have produced manifest duality actions for Born-Infeld theory [9]).

We will now go to the case where $\tau = i$ as this will ease our calculation greatly. We will reinstate the dilaton coupling later. We will now follow the method of [1] Schwarz and Sen to show that this action gives the Born-Infeld in 4-dimensions. First use gauge invariance to set $A_0 = 0$. Then, as discussed in [1] one of the A_i^L field becomes auxiliary and may be eliminated in favor of the other. Let us work in the concrete case where we will eliminate A^2 from the action (9). We find the equation of motion for A^2 by varying the action (9) (with M equal to the identity:

$$\vec{\nabla} \wedge (\vec{M}(B^1, B^2) - \vec{E}^1) = 0$$

Where

$$\vec{M}(B^1, B^2) = \frac{\vec{B}^2 - (\vec{B}^2 \cdot \vec{B}^1) \vec{B}^1 + (\vec{B}^1 \cdot \vec{B}^1) \vec{B}^2}{\sqrt{1 + (\vec{B}^1)^2 (\vec{B}^2)^2 - (\vec{B}^1 \cdot \vec{B}^2)^2 + (\vec{B}^1)^2 + (\vec{B}^2)^2}}$$

with \vec{B} being a vector in 3 dimensions. We can solve this by writing

$$\vec{M}(B^1, B^2) - \vec{E}^1 = \vec{\nabla} \psi$$

We still have some gauge symmetry left $\delta \vec{A}^1 = \vec{\nabla} \chi$ to eliminate $\nabla \psi$. Leaving the equation:

$$\vec{M}(B^1, B^2) - \vec{E}^1 = 0 \quad (17)$$

The equivalent equation in Schwarz Sen approach to Maxwell theory is simply $\vec{B}^2 = \vec{E}^1$, which greatly facilitates the calculation and explicitly shows that the pair of Electric and Magnetic fields are related by duality.

The next step is to solve this equation for \vec{B}^2 . After some manipulations we find

$$\vec{B}^2 = \frac{\vec{E}^1 + (\vec{E}^1 \cdot \vec{B}^1) \vec{B}^1}{\sqrt{1 + (\vec{B}^1)^2 (\vec{B}^2)^2 - (\vec{B}^1 \cdot \vec{B}^2)^2 + (\vec{B}^1)^2 + (\vec{B}^2)^2}} \quad (18)$$

As a simple check we can see that this equation for B^2 reduces to to Maxwell case to first order in fields.

We now substitute this into the action (9) and find:

$$S = \int_{M^4} d^4x \sqrt{-\eta'} \sqrt{(1 + (\vec{B}^1)^2 - (\vec{E}^1)^2 - (\vec{E}^1 \cdot \vec{B}^1)^2)}$$

This becomes after rewriting in terms of a four dimensional determinant:

$$S = \int_{M^4} d^4x \sqrt{-\det(\eta'_{\mu\nu} + F_{\mu\nu})} \quad (19)$$

This is of course the Born-Infeld with trivial background fields. If we reinstate the dilaton coupling and repeat the above procedure we see that we get the expected dilaton dependence. That is, we recover the action given in equation (1) without the axion term. We have so far been unable to repeat the process with the axion term, essentially because we have not been able to solve the analogue of equation (17) once the axion is included. There is no reason to believe that it can not be done and this detail would not add anything to the overall picture.

We generate the dual theory by repeating the process but instead we integrate out A^1 instead of A^2 . This gives the same action but with the expected dilaton inversion. So in this description the duality is a symmetry of the action (16). The two duality related theories are given by eliminating different fields from this action. It is a nice check that the two routes, one with da in the compact space and one with da in space time give (as they obviously should) the same results.

Discussion

We can interpret our results in the context of M-theory as follows. We will work with bosonic branes. To interpret the actions presented here as branes we write the metric as a pull-back onto the brane from the background space-time. That is:

$$G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN} \quad (20)$$

Where X^M are D dimensional spacetime coordinates, with g_{MN} the D-dimensional spacetime metric. $\partial_\mu = \frac{\partial}{\partial x^\mu}$ where x^μ are coordinates in the d-brane, $\mu = 0..d$. The dimensional reduction described above becomes double dimensional reduction in the brane picture. That is, we identify the compact brane coordinate with the compact space-time coordinate. Now we would like to justify our ansätze used for the dimensional reduction from this point of view. There are two spacetime U(1) gauge fields from Kaluza-Klein on the torus. The three-brane momentum in the compact dimensions couples to these gauge fields. By taking the

zero modes on the torus we are identifying the sector of the theory in which the 3-brane is neutral with respect to these fields. Hence, the truncation of these fields.

By not considering the scalar fields and two form fields in our ansatz for B we are truncating the part that is coupled to the area of the torus. (As H is self dual these fields will be duality related). We justify this as follows.

We wish to compare our double dimensional reduction from 11 to 9 dimensions of a 5-brane with the direct dimensional reduction of a 3-brane from 10 to 9 dimensions. Direct reduction implies that we do not wrap the brane around the compact space-time dimension. This induces an additional scalar field in the brane with a coupling given by the radius squared of the compact dimension. In the spirit of [10] the world volume dual of this field will then be identified with the two form potential B in the 5-brane picture. In doing this identification we must identify the radius, R of the 10th dimension with $\frac{1}{\sqrt{V}}$. With this identification can then see that our metric η is scaled by $\frac{1}{R}$. This is what is to be expected following arguments given in [11] to allow η' to be identified with the 10-dimensional metric in the Einstein frame (it is in this metric that the IIB 3-brane is self dual). If we consider the limit where the area of the torus goes to zero (keeping the complex structure fixed) we must go to the limit where R goes to infinity and lift to a non compact 10-dimensional theory. Hence, by truncating the fields that couple to the area of the torus we are going immediately to the description of the 3-brane in 10 dimensions. The supersymmetric version of these actions will be interpreted as the IIB D-3 brane from the double dimensional reduction of the M 5-brane on a torus. The IIB S-duality then becomes manifest as the modular group of the torus, as reported in [12]. Our S-duality in the brane is then a result of the IIB S-duality. We will report in the future the more precise M-theory, brane picture that extends the above ansatz and includes the important Ramond-Ramond fields.

Conclusions

We confirm that the duality related Born-Infeld theories with coupling to a background dilaton-axion can come naturally, from compactification of a self dual non-linear two form theory in 6 dimensions. The axion-dilaton becomes identified with the complex structure of the torus, and the duality symmetry then becomes associated with the modular group of the torus. This is as expected from the M-theory point of view where the one imagines the 3 brane as coming from wrapping the 5-brane on a torus.

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